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Bounds for the Minimum Numbers of Generators of Generalized Cohen–Macaulay Ideals

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1. INTRODUCTION AND MAIN RESULTS

Throughout this paper, R denotes a local ring with maximal ideal $\mathfrak{m}$, I an ideal of R, and S the factor ring R/I.

We call I a generalized Cohen–Macaulay ideal if S is a generalized Cohen–Macaulay ring. By definition [4, (3.1)], a local ring A with maximal ideal $\mathfrak{p}$ is a generalized Cohen–Macaulay ring if the ith local cohomology module $H^i_{\mathfrak{p}}(A)$ of A with support $\mathfrak{p}$ is of finite length for $i = 0,\dots, \dim A - 1$ (A is a Cohen–Macaulay ring iff $H^i_{\mathfrak{p}}(A) = 0$ for $i = 0,\dots, \dim A - 1$). The class of generalized Cohen–Macaulay local rings is rather large. For instance, if A is a factor ring of a Cohen–Macaulay ring, A being a generalized Cohen–Macaulay ring means that A is equidimensional and $A_{\mathfrak{p}}$ is a Cohen–Macaulay ring for all prime ideals $\mathfrak{q} \neq \mathfrak{p}$ of A [4, (2.5) and (3.8)].

The aim of this paper is to give bounds for the minimum number $v(I)$ of generators of a generalized Cohen–Macaulay ideal I in terms of the following invariants of R and S: $m = v(\mathfrak{m})$, $n = v(\mathfrak{m}S)$, $r = \dim R$, $s = \dim S$, the multiplicities $e(R)$, $e(S)$, and $l(H^i_{\mathfrak{m}S}(S))$, $i = 0,\dots, s-1$, where l denotes the length function.

Our interest in such bounds for $v(I)$ arose from the article [7] of Sally. There she proved that if R and S are Cohen–Macaulay rings, then $v(I) \leqslant e(R)\, e(S)^{r-s-1} + r - s - 1$. First, Sally's result may be generalized as

THEOREM 1. *Let R and S be generalized Cohen–Macaulay rings with $r > s$. Then*

$$v(I) \leqslant e(R)\left[e(S) + \sum_{i=0}^{s-1} \binom{s-1}{i} l(H^i_{\mathfrak{m}S}(S))\right]^{r-s-1} + r - s - 1$$
$$+ \sum_{i=0}^{r-1} \binom{r-1}{i} l(H^i_{\mathfrak{m}}(R)) + \sum_{i=0}^{s-1} \sum_{j=0}^{i} \binom{i}{j} l(H^j_{\mathfrak{m}S}(S)).$$

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If S is moreover an equicharacteristic local ring with $s > 1$, one can replace the term $e(S) + \sum_{i=0}^{s-1} \binom{s-1}{i} l(H^i_{\mathfrak{m}S}(S))$ by $e(S) + l(H^0_{\mathfrak{m}S}(S))$.

However, we must point out that in general, if $r-s-1$ is large, Theorem 1 does not yield a good bound for $v(I)$. In view of this fact one may ask about a bound for $v(I)$ which depends linearly on $e(S)$. Concerning this question, we have the following result which holds for an arbitrary local ring R.

THEOREM 2. *Let S be a generalized Cohen–Macaulay ring. Then*

$$\begin{aligned} v(I) \leqslant m - n &+ \sum_{i=0}^{s-1} \sum_{j=0}^{i} \binom{i}{j} l(H^j_{\mathfrak{m}S}(S)) \\ &+ (n-s-1)\left[e(S) + \sum_{i=0}^{s-1} \binom{s-1}{i} l(H^i_{\mathfrak{m}S}(S))\right] \\ &- \binom{n-s}{2} + 1. \end{aligned}$$

There are several interesting applications, see Section 2. The proofs of the theorems will be found in Section 3.

2. Applications and Remarks

2.1

Let R be a generalized Cohen–Macaulay ring. Then, applying Theorem 1 to the ideal $I = \mathfrak{m}^t$, we have

$$l(\mathfrak{m}^t/\mathfrak{m}^{t+1}) \leqslant e(R)\, t^{r-1} + r - 1 + \sum_{i=0}^{r-1} \binom{r-1}{i} l(H^i_{\mathfrak{m}}(R))$$

for all $t > 0$ (cf. [7, Corollary 1.3]). In particular, for $t = 1$ we obtain a bound for the embedding dimension m of R which generalizes a similar result of Abhyankar for local Cohen–Macaulay rings [1, (1)]. Further, by the main theorem of Eakin and Sathaye in [5] one can also use the above bound for $l(\mathfrak{m}^t/\mathfrak{m}^{t+1})$ to estimate the least integer t such that there exists a system of parameters $x_1, \ldots, x_r$ of R with $(x_1, \ldots, x_r)\, \mathfrak{m}^{t-1} = \mathfrak{m}^t$ (cf. [7, Theorem 2.2]).

2.2

Let C be an arbitrary curve in the three-dimensional projective space $\mathbb{P}^3_k$ over a field k and I_C the defining prime ideal of C in the polynomial ring

$k[x_1, x_2, x_3, x_4]$. Let A denote the local ring of the vertex of the affine cone over C and $\mathfrak{p}$ the maximal ideal of A. Then $A = k[x_1, x_2, x_3, x_4]_{(x_1,x_2,x_3,x_4)}/(I_C)$ is a two-dimensional local domain, hence a generalized Cohen–Macaulay ring with $H^0_{\mathfrak{p}}(A) = 0$. Thus, applying the second part of Theorem 1 we get the following bound for the minimum number $v(I_C)$ of generators of I_C:

$$v(I_C) \leqslant e(A) + l(H^1_{\mathfrak{p}}(A)) + 1. \tag{1}$$

If C does not lie in any plane of $\mathbb{P}^3_k$ (otherwise we always have $v(I_C) = 2$), Theorem 2 yields

$$v(I_C) \leqslant e(A) + 2l(H^1_{\mathfrak{p}}(A)). \tag{2}$$

Clearly, (1) gives a better bound for $v(I_C)$ than (2) except when C is an arithmetically Cohen–Macaulay curve (i.e., A is a Cohen–Macaulay ring), see 2.3.

From a result of Becker [2, Theorem] we can also deduce that

$$v(I_C) \leqslant e(A)^2 + 1. \tag{3}$$

In general, one could not compare (1) with (3). However, if C is an arithmetically Buchsbaum curve, i.e., if $H^1_{\mathfrak{p}}(A)$ is a vector space over k (see, e.e., [3, Sect. 2]), (1) gives a better bound for $v(I_C)$ than (3). Indeed, in this case we know by [3, Lemma 2] that

$$v(I_C) \geqslant 3l(H^1_{\mathfrak{p}}(A)) + 1. \tag{4}$$

A comparison between (1) and (4) shows that $e(A) \geqslant 21(H^1_{\mathfrak{p}}(A))$. Hence we can replace $l(H^1_{\mathfrak{p}}(A))$ of (1) by $\frac{1}{2}e(A)$ and get

$$v(I_C) \leqslant \tfrac{3}{2}e(A) + 1. \tag{5}$$

EXAMPLE. Let $C \subset \mathbb{P}^3_k$ be the curve given parametrically by $\{t_1^2, t_1^3, t_1t_2, t_2\}$. It is not hard to see that there exists an exact sequence of A-modules of the form

$$0 \to A \to k[t_1, t_2]_{(t_1,t_2)} \to k \to 0.$$

Note that $k[t_1, t_2]_{(t_1,t_2)}$ is a Cohen–Macaulay A-module so that $H^i_{\mathfrak{p}}(k[t_1, t_2]_{(t_1,t_2)}) = 0$ for $i = 0, 1$. Then, from the above sequence we can deduce that $H^1_{\mathfrak{p}}(A) \cong k$. Thus, C is an arithmetically Buchsbaum curve with $l(H^1_{\mathfrak{p}}(A)) = 1$. Further, it is also easily seen that $e(A) = 2$. Setting these data in (1) and (5) we get $v(I_C) = 4$, hence equality in (1), (2), (4), (5).

2.3

Let C now be an arithmetically Cohen–Macaulay variety of codimension 2 in the n-dimensional projective space $\mathbb{P}_k^n$ over a field k, $n \geqslant 3$, which does not lie in any hyperplane of $\mathbb{P}_k^n$. Then, with the corresponding notations I_C, A, $\mathfrak{p}$ as in 2.2, we can deduce from Theorem 2 that $v(I_C) \leqslant e(A)$, while Theorem 1 and known results of Sally [7, Theorem 2.1] and Rees [6] only yield $v(I_C) \leqslant e(A) + 1$.

EXAMPLE. Let $C \subset \mathbb{P}_k^3$ be the Veronese curve given parametrically by $\{t_1^3, t_1^2 t_2, t_1 t_2^2, t_2^3\}$. It is well known that C is an arithmetically Cohen–Macaulay curve with $v(I_C) = e(A) = 3$.

2.4

The bound for $v(I)$ in Theorem 2 is very simple if $n = s$, $s + 1$.

If $n = s$, i.e., S is a regular ring, we have $v(I) \leqslant m - n$. Since m is always bounded above by $v(I) + n$, we can conclude that $v(I) = m - n$ in this case.

If $n = s + 1$, we will first see that $H^i_{\mathfrak{m}S}(S) = 0$ for $i = 1,\dots, s - 1$. Set $S_1 = S/\bigcup_{t=1}^{\infty} O_S : \mathfrak{m}^t S$ and consider the exact sequence

$$0 \to \bigcup_{t=1}^{\infty} O_S : \mathfrak{m}^t S \to S \to S_1 \to 0.$$

Since $\bigcup_{t=1}^{\infty} O_S : \mathfrak{m}^t S = H^0_{\mathfrak{m}S}(S)$, from this sequence we can deduce that $H^0_{\mathfrak{m}S}(S_1) = 0$, $H^i_{\mathfrak{m}S}(S_1) \cong H^i_{\mathfrak{m}S}(S)$ for $i = 1,\dots, s - 1$. Therefore, S_1 is a generalized Cohen–Macaulay ring with depth $S_1 \geqslant 1$. Let S_1^* denote the completion of S_1 with respect to its maximal ideal. Since S_1^* is faithfully flat over S_1, S_1^* is also a generalized Cohen–Macaulay ring with depth $S_1^* \geqslant 1$. It follows by [4, (2.5)(ii) and (3.4)] that the zeroideal of S_1^* is unmixed. By Cohen's theorem, S_1^* is then the factor ring of a n-dimensional regular local ring by an unmixed ideal of height 1. Since such an ideal is principal, S_1^* is a Cohen–Macaulay ring, and so is S_1. Thus, $H^i_{\mathfrak{m}S}(S) \cong H^i_{\mathfrak{m}S}(S_1) = 0$ for $i = 1,\dots, s - 1$. Hence, by Theorem 2 we have

$$v(I) \leqslant m - n + sl(H^0_{\mathfrak{m}S}(S)) = m + s[l(H^0_{\mathfrak{m}S}(S)) - 1] - 1.$$

3. Proofs of the Main Results

LEMMA 1. *The following statements hold for an element $x \in \mathfrak{m}$:*

(i) $v(I) \leqslant l((I, x)/(\mathfrak{m}I, x)) + l(I : x/I)$,

(ii) $v(I) \leqslant l((I, x)/\mathfrak{m}(I, x)) - 1 + l(I : x/I)$ *if* $x \notin \mathfrak{m}I$,

(iii) $v(I) \leqslant l(R/(x))$ *if I is an $\mathfrak{m}$-primary ideal.*

Proof. For (i) we have

$$v(I) = l(I/mI) = l(I/(mI, x(I:x)) + l(mI, x(I:x))/mI).$$

Note that $I/(mI, x(I:x)) = I/I \cap (mI, x) \cong (I, x)/(mI, x)$ and that $(mI, x(I:x))/mI \cong x(I:x)/x(I:x) \cap mI \cong I:x/mI:x$. Then

$$\begin{aligned} v(I) &= l((I,x)/(mI,x)) + l(I:x/mI:x) \\ &\leqslant l((I,x)/(mI,x)) + l(I:x/I). \end{aligned}$$

(ii) is only a consequence of (i) because $l((I,x)/(mI,x)) = l((I,x)/m(I,x)) - 1$ if $x \notin mI$.

For (iii) we may assume that $l(R/(x)) < \infty$. Then

$$\begin{aligned} v(I) &\leqslant l(I/xI) = l(R/(x)) + l((x)/xI) - l(R/I) \\ &= l(R/(x)) + l(R/(I, 0:x) - l(R/I) \\ &\leqslant l(R/(x)). \end{aligned}$$

Proof of Theorem 1. Without restriction we may assume that the residue field R/m is infinite. For R may be replaced by the local ring $R[u]_{mR[u]}$, where u is an indeterminate over R, and I by the ideal $IR[u]_{mR[u]}$. It is easy to see that these changes will never cause us any problem. With this assumption, by [7, Lemma 1.1] and the proof of [8, Theorem 22, Sect. 10, Chap. VIII] we can find a system of parameters $x_1,..., x_r$ of R such that $x_1,..., x_s$ form a system of parameters for S, $e(x_1,..., x_s; S) = e(S)$, and $e(x_1,..., x_r; R) = e(R)$, where $e(x_1,..., x_s; S)$ and $e(x_1,..., x_r; R)$ denote the multiplicities of S and R relative to the ideals $(x_1,..., x_s)S$ and $(x_1,..., x_r)$, respectively. By the help of this system of parameters $x_1,..., x_r$ of R we are now going to estimate $v(I)$.

First, applying Lemma 1(i) and [4, (3.5)] we have

$$\begin{aligned} v(I) &\leqslant l((I, x_1,..., x_s)/(mI, x_1,..., x_s)) \\ &\quad + \sum_{i=1}^{s} l((I, x_1,..., x_{i-1}): x_i/(I, x_1,..., x_{i-1})) \\ &\leqslant l((I, x_1,..., x_s)/(mI, x_1,..., x_s)) \\ &\quad + \sum_{i=0}^{s-1} \sum_{j=0}^{i} \binom{i}{j} l(H^j_{m}(S)). \end{aligned}$$

Set $L = e(S) + \sum_{i=0}^{s-1} \binom{s-1}{i} l(H^i_{m}(S))$ and note that $l((R/(I, x_1,..., x_s)) \leqslant L$ [4, (3.7)]. Then, $x_i^L \in (I, x_1,..., x_s)$ for $i = s+1,..., r$. Hence

$$
\begin{aligned}
&l((I, x_1, \dots, x_s)/(\mathfrak{m}I, x_1, \dots, x_s)) \\
&\quad \leqslant l((I, x_1, \dots, x_s)/(\mathfrak{m}I, x_1, \dots, x_s, x_{s+1}^L, \dots, x_{r-1}^L)) \\
&\quad + r - s - 1.
\end{aligned}
$$

By Lemma 1(iii) and [4, (3.7)] we have

$$
\begin{aligned}
&l((I, x_1, \dots, x_s)/(\mathfrak{m}I, x_1, \dots, x_s, x_{s+1}^L, \dots, x_{r-1}^L)) \\
&\quad \leqslant l(R/(x_1, \dots, x_s, x_{s+1}^L, \dots, x_{r-1}^L, x_r)) \\
&\quad \leqslant e(R)\, L^{r-s-1} + \sum_{i=0}^{r-1} \binom{r-1}{i} l(H_{\mathfrak{m}}^i(R)).
\end{aligned}
$$

Therefore we can conclude that

$$
\begin{aligned}
v(I) \leqslant e(R)\, L^{r-s-1} &+ \sum_{i=0}^{r-1} \binom{r-1}{i} l(H_{\mathfrak{m}}^i(R)) \\
&+ r - s - 1 + \sum_{i=0}^{s-1} \sum_{j=0}^{i} \binom{i}{j} l(H_{\mathfrak{m}S}^j(S)).
\end{aligned}
$$

We have proved the first part of Theorem 1. For the second part of Theorem 1 it suffices to show that $x_i^l \in (I, x_1, \dots, x_s)$ for $i = s+1, \dots, r$, where $l := e(S) + l(H_{\mathfrak{m}S}^0(S))$.

First, without restriction we may assume that R is a complete local ring, see [2, Remark 1.4]. Set $S_1 = R/\bigcup_{t=1}^{\infty} I : \mathfrak{m}^t = S/\bigcup_{t=1}^{\infty} O_S : \mathfrak{m}^t S$. As in 2.4 we can see that the zeroideal of S_1 is unmixed. Further, since $\bigcup_{t=1}^{\infty} O_S : \mathfrak{m}^t S$ is a S-module of finite length, $x_1, \dots, x_s$ still form a system of parameters for S_1 and, since $\dim S_1 = s > 1$, the multiplicity of S_1 relative to the ideal $(x_1, \dots, x_s) S_1$ is equal to $e(S)$. Therefore, since S is an equicharacteristic local ring, we can use the proof of [2, Proposition 1.5] to show that $x_i^{e(S)} \in (\bigcup_{t=1}^{\infty} I : \mathfrak{m}^t, x_1, \dots, x_s)$ for $i = s+1, \dots, r$.

If $x_i^l \notin (I, x_1, \dots, x_s)$ for some $i = s+1, \dots, r$, we must have $l > e(S)$, i.e., $H_{\mathfrak{m}S}^0(S) \neq 0$ because $H_{\mathfrak{m}S}^0(S) \cong \bigcup_{t=1}^{\infty} I : \mathfrak{m}^t / I$. It follows that

$$
(I, x_1, \dots, x_s, x_i^{e(S)}) \supset \cdots \supset (I, x_1, \dots, x_s, x_i^l) \supset (I, x_1, \dots, x_s)
$$

is a properly descending chain of ideals in $(\bigcup_{t=1}^{\infty} I : \mathfrak{m}^t, x_1, \dots, x_s)$. Hence

$$
\begin{aligned}
l(H_{\mathfrak{m}S}^0(S)) = l - e(S) &< l\left(\left(\bigcup_{t=1}^{\infty} I : \mathfrak{m}^t, x_1, \dots, x_s\right) \Big/ (I, x_1, \dots, x_s)\right) \\
&\leqslant l\left(\bigcup_{t=1}^{\infty} I : \mathfrak{m}^t / I\right) = l(H_{\mathfrak{m}S}^0(S)),
\end{aligned}
$$

a contradiction.

Proof of Theorem 2. Let $x_1,\dots, x_n$ be elements in R such that they form a basis for $\mathfrak{m}S$ in S, i.e., $(I, x_1,\dots, x_n) = \mathfrak{m}$. Then $x_i \notin (I, x_1,\dots, x_{i-1})$ for all $i = 1,\dots, n$. Hence, applying Lemma 1(ii) we have

$$v(I) \leqslant m - n + \sum_{i=1}^{n} l((I, x_1,\dots, x_{i-1}) : x_i/(I, x_1,\dots, x_{i-1})).$$

It remains to estimate $l((I, x_1,\dots, x_{i-1}) : x_i/(I, x_1,\dots, x_{i-1}))$ for $i = 1,\dots, n$. For that we may assume as in the proof of Theorem 1 that the residue field $R/\mathfrak{m}$ is infinite. Then, by the proof of [8, Theorem 22, Sec. 10, Chap. VIII], we can choose $x_1,\dots, x_s$ such that they form a system of parameters for S with $e(x_1,\dots, x_s; S) = e(S)$. It follows by [4, (3.5)] that

$$\begin{aligned} &l((I, x_1,\dots, x_{i-1}) : x_i/(I, x_1,\dots, x_{i-1})) \\ &\qquad \leqslant \sum_{j=0}^{i-1} \binom{i-1}{j} l(H^j_{\mathfrak{m}S}(S)) \end{aligned}$$

for $i = 1,\dots, s$. For $i = s + 1,\dots, n - 1$ we have

$$\begin{aligned} &l((I, x_1,\dots, x_{i-1}) : x_i/(I, x_1,\dots, x_{i-1})) \\ &\quad \leqslant l(\mathfrak{m}/(I, x_1,\dots, x_{i-1})) \\ &\quad = l(\mathfrak{m}/(I, x_1,\dots, x_s)) - l((I, x_1,\dots, x_{i-1})/(I, x_1,\dots, x_s)) \\ &\quad \leqslant l(R/(I, x_1,\dots, x_s)) - 1 - (i - s - 1) \\ &\quad \leqslant e(S) + \sum_{i=0}^{s-1} \binom{s-1}{i} l(H^i_{\mathfrak{m}S}(S)) - i + s \qquad [4, (3.7)]. \end{aligned}$$

Note that $l((I, x_1,\dots, x_{n-1}) : x_n/(I, x_1,\dots, x_{n-1})) = 1$. Then

$$\begin{aligned} v(I) \leqslant m - n &+ \sum_{i=0}^{s-1} \sum_{j=0}^{i} \binom{i}{j} l(H^j_{\mathfrak{m}S}(S)) \\ &+ (n - s - 1) \left[e(S) + \sum_{i=0}^{s-1} \binom{s-1}{i} l(H^i_{\mathfrak{m}S}(S)) \right] \\ &- \binom{n-s}{2} + 1. \end{aligned}$$

Remark. If S is moreover an equicharacteristic local ring with $s > 1$, we can estimate $l((I, x_1,\dots, x_{i-1}) : x_i/(I, x_1,\dots, x_{i-1}))$, $i = s + 1,\dots, n$, in another way. First, set $l = e(S) + l(H^0_{\mathfrak{m}S}(S))$. Then, as in the proof for the second part of Theorem 1, we have $x_j^l \in (I, x_1,\dots, x_{i-1})$ for $j = i,\dots, n$. Thus, if we denote by J the set of all tuples $(a) = (a_{i+1},..., a_n)$ of $n-i$ integers $a_{i+1},..., a_n$ with $0 \leqslant a_{i+1},..., a_n < 1$, every element of R may be written in the form

$$\sum_{(a)\in J} y_{(a)} x_i^{k(a)} x_{i+1}^{a_{i+1}} \cdots x_n^{a_n} + g$$

for some $k(a) \geqslant 0$, $y_{(a)} \in (R \backslash \mathfrak{m}) \cup \{0\}$ and $g \in (I, x_1,..., x_{i-1})$. Order J somehow completely. Choose, for every element $(b) \in J$, an element $f_{(b)}$ from the set of all elements in $(I, x_1,..., x_{i-1}): x_i$ having the above form with $y_{(b)} \neq 0$, $y_{(a)} = 0$ for all $(a) < (b)$ such that $k(b)$ is as small as possible. It is easily seen that there exist, for every element $f \in (I, x_1,..., x_{i-1}): x_i$, elements $z_{(b)} \in (R \backslash \mathfrak{m}) \cup \{0\}$ and $g \in (I, x_1,..., x_{i-1})$ such that

$$f - \sum_{(b)\in J} z_{(b} f_{(b)} \in (I, x_1,..., x_{i-1}).$$

From this fact we can then conclude that

$$l((I, x_1,..., x_{i-1}): x_i/(I, x_1,..., x_{i-1})) \leqslant l^{n-i}.$$

By the proof of Theorem 2, this estimation leads to

THEOREM 3. *Let S be an equicharacteristic local generalized Cohen–Macaulay ring with $s > 1$. Then*

$$v(I) \leqslant m - n + \sum_{i=0}^{s-1} \sum_{j=0}^{i} \binom{i}{j} l(H_{\mathfrak{m}S}^j(S)) + \sum_{i=0}^{n-s-1} [e(S) + l(H_{\mathfrak{m}S}^0(S))]^i.$$

For the minimum number of generators of the defining prime ideal of a curve $C \subset \mathbb{P}^3$ which does not lie in any plane of $\mathbb{P}^3$ Theorem 3 gives the bound (1) of Section 2, 2.2 again.

Finally, we emphasize that if S is an equicharacteristic local generalized Cohen–Macaulay ring with $s > 1$, by combining the different estimations of $l((I, x_1,..., x_{i-1}): x_i/(I, x_1,..., x_{i-1}))$, $i = s+1,..., n$, in the proof of Theorem 2 and in the remark, we can obtain bounds for $v(I)$ which are "polynomials" in $e(S)$ of any degree between 1 and $n-s-1$.

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